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On the Hyers–Ulam Stability of ψ-Additive Mappings

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Let E_1 be a real normed vector space and E_2 a real Banach space. S. M. Ulam posed the problem: When does a linear mapping near an approximately additive mapping $f: E_1 \to E_2$ exist? We give a new generalized solution to Ulam's problem for ψ-additive mappings. Some relations with the asymptotic differentiability are also indicated.

INTRODUCTION

Let E_1 be a real normed vector space and E_2 a real Banach space.

Assume that $f: E_1 \to E_2$ is an approximately additive mapping. S. M. Ulam posed the problem: Give conditions in order for a linear mapping near an approximately additive mapping to exist [12, 13].

In 1941 D. H. Hyers [6] considered approximately additive mappings $f: E_1 \to E_2$ satisfying

$$\|f(x+y)-f(x)-f(y)\| < \varepsilon \tag{1}$$

for all $x, y \in E_1$.

He proved that the limit

$$T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \tag{2}$$

exists for all $x \in E_1$ and that $T: E_1 \to E_2$ is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leqslant \varepsilon.$$

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No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed x, then the mapping T given by (2) is linear.

In 1978, a generalized solution to Ulam's problem for approximately linear mappings was given by Th. M. Rassias [9].

He considered a mapping $f: E_1 \to E_2$ satisfying the condition of continuity of $f(tx)$ in t for each fixed x and assumed the weaker condition

$$\|f(x+y)-f(x)-f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p), \qquad \text{for any} \quad x, y \in E_1, \tag{3}$$

where $\theta \geqslant 0$ and $0 \leqslant p < 1$.

He proved that the limit (2) exists for all $x \in E_1$ and that $T: E_1 \to E_2$ is the unique linear mapping satisfying

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{2-2^p} \|x\|^p.$$

The proof given in [9] works also when $p<0$. Th. M. Rassias [10] posed the problem whether such a theorem can also be proved for $p \geqslant 1$.

In [3] Z. Gajda followed a similar approach as in [9] and obtained a solution of this problem for $p>1$.

His result states that the mapping $T: E_1 \to E_2$ defined by

$$T(x) = \lim_{n \to \infty} 2^n f(2^{-n}x) \tag{4}$$

is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{2^p-2} \|x\|^p. \tag{5}$$

The problem when $p=1$ is still open (cf. R. Ger [4]). Let E_1 be a real normed vector space and E_2 a real Banach space. We introduce the following notion.

Definition. A mapping $f: E_1 \to E_2$ is ψ-additive if and only if there exist $\theta \geqslant 0$ and a function $\psi: R_+ \to R_+$ such that $\lim_{t \to \infty}(\psi(t)/t) = 0$ and

$$\|f(x+y)-f(x)-f(y)\| \leqslant \theta[\psi(\|x\|) + \psi(\|y\|)], \qquad \text{for all} \quad x, y \in E_1.$$

Theorem 1. *Consider E_1 to be a real normed vector space E_2 a real Banach space and let $f: E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x.*

If f is ψ-additive and ψ satisfies

(1) $\psi(ts) \leqslant \psi(t)\psi(s)$, *for all $t, s \in R_+$*

(2) $\psi(t) < t$, *for all $t > 1$*

then there exists a unique linear mapping $T: E_1 \to E_2$ *such that* $\|f(x) - T(x)\| \leqslant (2\theta/(2-\psi(2)))\, \psi(\|x\|)$, *for all* $x \in E_1$.

Proof. We will show that

$$\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \leqslant \left[\theta \sum_{m=0}^{n-1} \left(\frac{\psi(2)}{2} \right)^m \right] \psi(\|x\|) \tag{6}$$

for any positive integer n, and for all $x \in E_1$. The proof of (6) follows by induction on n. For $n = 1$ by ψ-additivity of f we have

$$\|f(2x) - 2f(x)\| \leqslant 2\theta\psi(\|x\|),$$

which implies

$$\left\| \frac{f(2x)}{2} - f(x) \right\| \leqslant \theta\psi(\|x\|).$$

Assume now that (6) holds for n and we want to prove it for the case $(n+1)$. Replacing x by $2x$ in (6) we obtain

$$\left\| \frac{f(2^n \cdot 2x)}{2^n} - f(2x) \right\| \leqslant \left[\theta \sum_{m=0}^{n-1} \left(\frac{\psi(2)}{2} \right)^m \right] \psi(2\,\|x\|).$$

Since $\psi(2\,\|x\|) \leqslant \psi(2)\,\psi(\|x\|)$ we get

$$\left\| \frac{f(2^{n+1} x)}{2^n} - f(2x) \right\| \leqslant \left[\theta \sum_{m=0}^{n-1} \left(\frac{\psi(2)}{2} \right)^m \right] \psi(2)\,\psi(\|x\|). \tag{7}$$

Multiplying both sides of (7) by 1/2 we obtain

$$\left\| \frac{f(2^{n+1} x)}{2^{n+1}} - \frac{f(2x)}{2} \right\| \leqslant \left[\theta \sum_{m=0}^{n} \left(\frac{\psi(2)}{2} \right)^m \right] \psi(\|x\|).$$

Now, using the triangle inequality we deduce

$$\begin{aligned} &\left\| \frac{1}{2^{n+1}} [f(2^{n+1} x)] - f(x) \right\| \\ &\quad \leqslant \left\| \frac{1}{2^{n+1}} [f(2^{n+1} x)] - \frac{1}{2} [f(2x)] \right\| + \left\| \frac{1}{2} [f(2x)] - f(x) \right\| \\ &\quad \leqslant \left[\theta \sum_{m=1}^{n} \left(\frac{\psi(2)}{2} \right)^m \right] \psi(\|x\|) + \theta\psi(\|x\|) \\ &\quad = \theta\psi(\|x\|) \left[1 + \sum_{m=1}^{n} \left(\frac{\psi(2)}{2} \right)^m \right], \qquad \text{which proves (6).} \end{aligned}$$

Thus,

$$\left\| \frac{1}{2^{n+1}} [f(2^{n+1}x)] - f(x) \right\| \leqslant \theta\psi(\|x\|) \left[1 + \sum_{m=1}^{\infty} \left(\frac{\psi(2)}{2} \right)^m \right]$$
$$\leqslant \frac{2\theta\psi(\|x\|)}{2 - \psi(2)}.$$

For $m > n > 0$ we have

$$\left\| \frac{1}{2^m} [f(2^m x)] - \frac{1}{2^n} [f(2^n x)] \right\|$$
$$= \frac{1}{2^n} \left\| \frac{1}{2^{m-n}} [f(2^m x)] - [f(2^n x)] \right\|$$
$$= \frac{1}{2^n} \left\| \frac{1}{2^r} [f(2^r y)] - f(y) \right\|, \qquad \text{where} \quad r = m - n \text{ and } y = 2^n x.$$

Hence,

$$\left\| \frac{1}{2^m} [f(2^m x)] - \frac{1}{2^n} [f(2^n x)] \right\|$$
$$\leqslant \frac{1}{2^n} \theta \left[\frac{2\psi(\|y\|)}{2 - \psi(2)} \right] = \frac{1}{2^n} \theta \left[\frac{2\psi(2^n \|x\|)}{2 - \psi(2)} \right]$$
$$\leqslant \frac{1}{2^n} \theta \left[\frac{2(\psi(2^n))\, \psi(\|x\|)}{2 - \psi(2)} \right] \leqslant \left(\frac{\psi(2)}{2} \right)^n \theta \frac{2\psi(\|x\|)}{2 - \psi(2)}.$$

But since $\lim_{n \to \infty} (\psi(2)/2)^n = 0$, we have that $\{(1/2^n)[f(2^n x)]\}_{n \in N}$ is a Cauchy sequence.

Set $T(x) = \lim_{n \to \infty} (1/2^n)[f(2^n x)]$, for all $x \in E_1$.

We will prove that T is additive. For this

$$\|f[2^n(x+y)] - f(2^n x) - f(2^n y)\| \leqslant \theta[\psi(\|2^n x\|) + \psi(\|2^n y\|)]$$
$$= \theta[\psi(2^n \|x\|) + \psi(2^n \|y\|)] \leqslant \theta\psi(2^n)[\psi(\|x\|) + \psi(\|y\|)],$$

which implies that

$$(1/2^n)\, \|f[2^n(x+y)] - f(2^n x) - f(2^n y)\| \leqslant (\psi(2^n)/2^n)\, \theta[\psi(\|x\|) + \psi(\|y\|)]$$
$$\leqslant (\psi(2)/2)^n\, \theta[\psi(\|x\|) + \psi(\|y\|)].$$

However, $\lim_{n \to \infty} (\psi(2)/2)^n = 0$, thus

$$\lim_{n \to \infty} \frac{1}{2^n} \|f[2^n(x+y)] - f(2^n x) - f(2^n y)\| = 0.$$

Therefore

$$T(x+y)=T(x)+T(y), \qquad \text{for all} \quad x, y \in E_1. \tag{8}$$

Because of (8) it follows that $T(rx)=rT(x)$ for any rational number r.

Using the same argument as in [9], we obtain that $T(ax)=aT(x)$ for any real value of a. Hence T is a linear mapping.

From $\|(1/2^n)f(2^n x)-f(x)\| \leqslant 2\theta\psi(\|x\|)/(2-\psi(2))$ taking the limit as $n \to \infty$ we obtain

$$\|T(x)-f(x)\| \leqslant \frac{2\theta\psi(\|x\|)}{2-\psi(2)}. \tag{9}$$

Claim that T is the unique such linear mapping.

Suppose that there exists another one, denoted by $g: E_1 \to E_2$, satisfying

$$\|f(x)-g(x)\| \leqslant \frac{2\theta_1\psi_1(\|x\|)}{2-\psi_1(2)}. \tag{10}$$

From (9) and (10) we get

$$\begin{aligned}\|T(x)-g(x)\| &\leqslant \|T(x)-f(x)\| + \|f(x)-g(x)\| \\ &\leqslant \frac{2\theta\psi(\|x\|)}{2-\psi(2)} + \frac{2\theta_1\psi_1(\|x\|)}{2-\psi_1(2)}.\end{aligned}$$

Then,

$$\begin{aligned}\|T(x)-g(x)\| &= \left\|\frac{1}{n}\,T(nx)-\frac{1}{n}\,g(nx)\right\| \\ &\leqslant \frac{\psi(n)}{n}\,\frac{2\theta\psi(\|x\|)}{2-\psi(2)} + \frac{\psi_1(n)}{n}\,\frac{2\theta_1\psi_1(\|x\|)}{2-\psi_1(2)},\end{aligned}$$

for every positive integer $n > 1$.

However, $\lim_{n\to\infty}(\psi(n)/n)=0=\lim_{n\to\infty}(\psi_1(n)/n)$. Therefore $T(x)\equiv g(x)$ for all $x \in E_1$. Q.E.D.

Remarks. (1) If $\psi(t)=t^p$ with $0 \leqslant p < 1$ then from Theorem 1, we obtain the following result proved in [9].

THEOREM 2. *Consider E_1, E_2 to be two Banach spaces, and let $f: E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \theta[\|x\|^p + \|y\|^p], \qquad \textit{for any} \quad x, y \in E_1.$$

Then there exists a unique linear mapping $T: E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{2 - 2^p} \|x\|^p, \qquad \textit{for any} \quad x \in E_1.$$

(2) If $p < 0$ and

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ t^p & \text{if} \quad t > 0 \end{cases}$$

then from Theorem 1 we obtain a generalization of Theorem 2 for p a negative real number.

The mapping T defined by Theorem 1 has some remarkable properties.

(i) If $f(S)$ is bounded, where $S = \{x \in E_1 \mid \|x\| = 1\}$ then T is continuous.

Indeed, this is a consequence of the inequalities

$$\begin{aligned} \|T(x)\| &\leqslant \|f(x)\| + \|T(x) - f(x)\| \\ &\leqslant \|f(x)\| + \frac{2\theta}{2 - \psi(2)} \psi(\|x\|) \\ &\leqslant \|f(x)\| + \frac{2\theta}{2 - \psi(2)} \psi(1), \qquad \text{for all} \quad x \in S. \end{aligned}$$

(ii) In [8] the concept of an asymptotically linear operator is defined. A mapping $f: E_1 \to E_2$ is asymptotically linear if there exists a continuous linear operator $u: E_1 \to E_2$ such that $\lim_{\|x\| \to +\infty} (\|f(x) - u(x)\| / \|x\|) = 0$. In this case we say that u is the asymptotic derivative of f.

Thus, when the operator T defined by Theorem 1 is continuous we have that f is asymptotically linear and T is its asymptotic derivative. Indeed, we have

$$\lim_{\|x\| \to +\infty} \frac{\|f(x) - T(x)\|}{\|x\|} \leqslant \frac{2\theta}{2 - \psi(2)} \lim_{\|x\| \to +\infty} \frac{\psi(\|x\|)}{\|x\|} = 0.$$

The fact that T is the asymptotic derivative of f is very important since in this case T conserves some properties of f as, for example,

(a) if f is compact then T is compact [8],

(b) if f is an α-contraction with respect to a measure of noncompactness α, then T has the same property [7].

The asymptotic derivative is important in the study of fixed points and in the bifurcation theory [1, 2, 8].

Also, the fact that T is the asymptotic derivative of f implies that f is quasibounded in the sense of the definition given by Granas in [5].

(iii) Suppose now that $E_1 = E_2 = E$, where E is a Banach space ordered by a pointed closed convex cone $\mathbf{K}$.

If $\mathbf{K}$ is invariant with respect to f, that is, $f(x) \in \mathbf{K}$ for all $x \in \mathbf{K}$, then the operator T defined by Theorem 1 is positive, that is, $T(\mathbf{K}) \subseteq \mathbf{K}$ (in this case T is monotone increasing). Moreover, if in this case T is continuous we have that f is asymptotically linear with respect to $\mathbf{K}$ and T is its asymptotic derivative along $\mathbf{K}$.

Denote by $r(T)$ the spectral radius of T and suppose that $f(\mathbf{K}) \subseteq \mathbf{K}$.

If f is compact on every bounded set of $\mathbf{K}$, T is continuous, and $r(T) < 1$ then f has a fixed point $x_* \in \mathbf{K}$.

This result is a consequence of [2, Theorem 8.8, p. 94].

So, an interesting problem is to study the spectral radius of T.

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